UKMT

UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 2nd FEBRUARY 2006

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

1. **C** Note that $\frac{1}{4} = \frac{6}{24}$ and $\frac{1}{6} = \frac{4}{24}$. So $\frac{5}{24}$ is half-way between them.

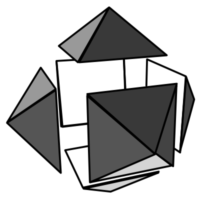
2. **B** The configuration includes three pairs of small rings. The rings in each pair are separate from the rings in the other two pairs so at least three rings will need to be cut. If the ring in each pair which is connected to the large ring is cut, then all of the rings can be separated. So the minimum number of rings which need to be cut is three.

3. **D** The values of the given options are 3, 7, 31, 63 and 127 respectively. All are prime, except 63. (*Note that* $2^c - 1$ *is never prime if c is composite. However, if p is prime,* $2^p - 1$ *is not necessarily prime. The smallest example of this occurs for* $p = 11$: $2^{11} - 1 = 23 \times 89$.)

4. **E** Whatever number is placed in the fifth box, the median and mode of the numbers will both be 7. For the mean to equal 7, the total of the 5 numbers must be 35. This means that the missing number is 9.

5. **B** Each pyramid has five faces, but the square base of each one is glued to the cube and therefore does not form a face of the star. So each pyramid contributes four faces to the star. As there are six such pyramids, the star has 24 faces.

(*It can be shown that the height of each of the pyramids is* $1/\sqrt{2}$. *If, and only if, the pyramids had been of height 1/2, however, then the angle between a triangular face of the pyramid and its square base would have been 45°. This would result in the solid having 12 faces rather than 24, as pairs of triangular faces would combine to form rhombi. The solid so formed is known as as a rhombic decahedron: a space-filling polyhedron.*)



(Diagram from http://dogfeathers.com/mark/rhdodec.html).

6. **A** Turbo will take 3 hours to complete the 12 miles, whilst Harriet will take 1 hour 30 minutes. So Harriet should set off 1 hour 30 minutes after Turbo, that is at 9:45 am.

7. **D** Statements A, B, and E may be made by the Queen of Spades, whether it is a day on which she is telling the truth or a day on which she is lying. She may also make statement C providing she is telling the truth that day. However, if she is telling the truth on a particular day then she could not make statement D since that would be a lie. Also, if she is not telling the truth on that day then she could not make statement D either, since that would then be a true statement. So she cannot make statement D.

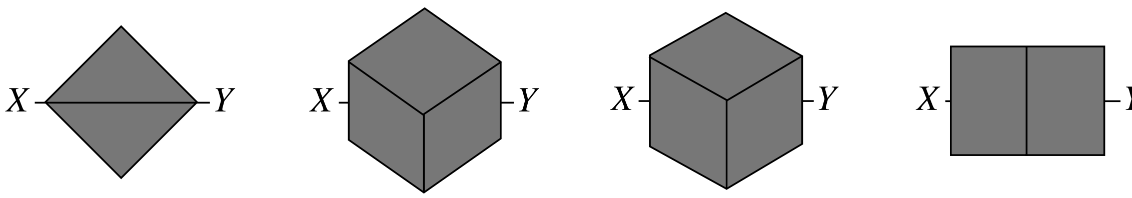
8. **C** The clock time in Melbourne when Sydney arrived was 32 hours (one complete day and 8 hours) ahead of the clock time in London when he left. So he arrived at 7:30 pm on Wednesday.

9. **D** Note that one cannot remove two touching coins; for if you did, any third coin originally making up a triangle with them would then be slideable. Note also that it is possible to remove the centre coin and each of the coins in a corner of the frame without enabling any of the remaining coins to slide. However, as each remaining coin touched at least one of the removed coins, it is not possible to remove any further coins. If, instead, a coin other than one of these four was removed first, then it would be one of the middle pair of an edge. Four coins would have touched this coin, so that would leave five possible coins to remove. Four of the five coins would lie along an edge of the frame, with two of these forming a triangle with the fifth. Clearly, only one of the three coins which form a triangle may now be removed, together with only one of the other two coins as they are touching. So we conclude that the maximum number of coins which may be removed is four.

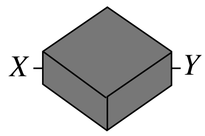
10. B The lunch bill and tip total £28, so Gill and her friend should pay £14 each. As Gill has paid £25.50, she should now receive £25.50 – £14, that is £11.50.

11. C At 8 o'clock, the obtuse angle between the hands of the clock is 120°. In the following six minutes, the minute hand turns though an angle of 36° whilst the hour hand turns through an angle of 3° in the same direction (clockwise!). So the obtuse angle between the hands increases by 33°.

12. D Three of the shadows are possible, the first, third and fourth. The diagrams below show four positions obtained by rotating a cube about XY, a line through the midpoints of a pair of opposite edges. Three of the shadows in the question correspond to three of these positions. Though the second shadow matches the angles and four of the lengths given by the second position, it does not match the other two lengths and so is not possible.

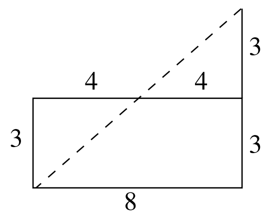


In fact, the second shadow corresponds to a cuboid which is half a cube, as shown alongside.



13. E Note that $x\%$ of y = $y\%$ of $x = xy/100$. So 2006% of 50 = 50% of 2006 = 1003.

14. D The diagram shows that the two pieces will fit together to form a right-angled triangle which has base 8 and height 6. The length of the hypotenuse = $\sqrt{6^2 + 8^2}$, that is 10, so the perimeter of the triangle is 24.



15. A The mean of $1.\dot{2}$ and $2.\dot{1}$ is $(1.\dot{2} + 2.\dot{1}) \div 2 = 3.\dot{3} \div 2 = 1.\dot{6}$.

16. B Al and Bertie have £55 between them, so Chris and Di have £150 – £55, that is £95, between them. As Al and Chris have £65 between them, the difference between the amounts Al and Di have is £95 – £65.

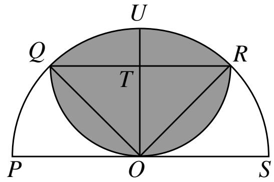
17. E The profit made would be £14 999.50, which is 29 999 times the original price. So that gives a profit of 29 999 × 100%, that is 2 999 900%.

18. E Note that $4^x + 4^x + 4^x + 4^x = 4 \times 4^x = 4^{x+1}$. So $x + 1 = 16$.

19. B Each interior angle of a regular pentagon is 108°, whilst each interior angle of a regular hexagon is 120°. The non-regular pentagon in the centre of the diagram contains two angles which are interior angles of the regular hexagon, two angles which are interior angles of the regular pentagon and a fifth angle, the one marked $x°$. So $x + 2 \times 120 + 2 \times 108 = 5 \times 108 = 540$. Hence $x = 84$.

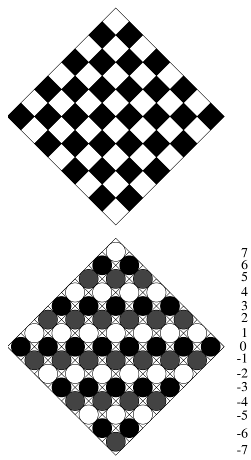
20. D As n is clearly odd, the series may be written as $1 + [-2 + 3] + [-4 + 5] + \ldots + [-(n-1) + n]$. So $1 - 2 + 3 - 4 + 5 - 6 + \ldots + (n-2) - (n-1) + n = 1 + 1 + \ldots + 1 = \frac{1}{2}(n+1)$. So $\frac{1}{2}(n + 1) = 2006$, that is $n = 4011$.

21. A Let T be the centre of the semicircle with diameter QR and let OT produced meet the circumference of the larger semicircle at U.



By symmetry, we note that OT is perpendicular to QR.
As $TR = TO = TQ$ (radii of semicircle), triangles ORT and OQT are both isosceles, right-angled triangles. So QOR is a right angle.
By Pythagoras' Theorem: $QR^2 = OQ^2 + OR^2 = 2^2 + 2^2 = 8$. So $QR = \sqrt{8} = 2\sqrt{2}$ and the radius of semicircle QOR is $\sqrt{2}$.
The area of the shaded region is equal to the area of semicircle QOR plus the area of the quadrant bounded by OQ, OR and arc QUR less the area of triangle OQR. So the required area is $\frac{1}{2}\pi(\sqrt{2})^2 + \frac{1}{4}\pi 2^2 - \frac{1}{2} \times 2 \times 2 = \pi + \pi - 2 = 2\pi - 2$.

22. B First note that counters of the same colour form diagonal lines across the board. The diagrams show the board before and after the counters are added. In both cases, the board has been rotated 45° anticlockwise. Note that the red counters are shown as grey.
Now consider the board to consist of 15 horizontal rows of squares, numbered from 7 to -7 as shown. The only rows in which the colour of the squares on the board matches the colour of the counters are rows 7, 6, 1, 0, -5 and -6. These contain 1, 2, 7, 8, 3 and 2 squares respectively, so the required fraction is 23/64.



23. B Triangles QPR and RPS are similar since $\angle QPR = \angle RPS$ and $\angle RQP = \angle SRP$. So $\frac{PR}{PS} = \frac{PQ}{PR}$. Hence $PR^2 = PQ \times PS = \frac{7}{3} \times \frac{48}{7} = 16$. So PR is 4 units long.
(*The geometric mean of x and y is defined to be* $\sqrt{xy}$, *so in this problem PR is the geometric mean of PQ and PS.*)

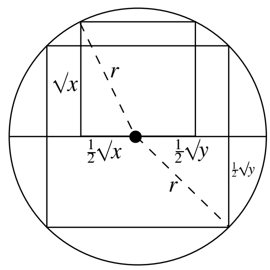
24. C Let r be the radius of the circle. Then, in the smaller square:

$$r^2 = (\sqrt{x})^2 + (\tfrac{1}{2}\sqrt{x})^2 = x + \tfrac{x}{4} = \tfrac{5x}{4}$$

and in the larger square:

$$r^2 = (\tfrac{1}{2}\sqrt{y})^2 + (\tfrac{1}{2}\sqrt{y})^2 = \tfrac{y}{4} + \tfrac{y}{4} = \tfrac{y}{2}$$

So $\frac{5x}{4} = \frac{y}{2}$ and we deduce that $x : y = 2 : 5$.



25. D First note that as j, l and m are all non-negative, the values of 5^j, 7^l and 11^m are all odd. However, the sum $5^j + 6^k + 7^l + 11^m$ is even, so we deduce that 6^k cannot be even and hence $k = 0$, that is $6^k = 1$. Now, for all positive integer values of j and m, the units digit of $5^j + 6^0 + 11^m$ is $5 + 1 + 1$, that is 7. So the units digit of 7^l is 9 and we deduce that $l = 2$ since 7, 49 and 343 are the only positive integer powers of 7 less than 2006. We now have $5^j + 6^0 + 7^2 + 11^m = 2006$, that is $5^j + 11^m = 1956$. The only positive integer powers of 11 less than 2006 are 11, 121 and 1331. These would require the value of 5^j to be 1945, 1835 and 625 respectively, and of these only 625 is a positive integer power of 5.
So $5^4 + 6^0 + 7^2 + 11^3 = 2006$.